Constrained Backward SDEs with Jumps: Application to Optimal Switching

Romuald ELIE
CEREMADE, CNRS, UMR 7534,
Université Paris-Dauphine,
and CREST
elie@ensae.fr

Idris KHARROUBI
LPMA, CNRS, UMR 7599,
Université Paris 7,
and CREST
kharroubi@ensae.fr

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Abstract

This paper enlarges the class of backward stochastic differential equation (BSDE) with jumps, adding some general constraints on all the components of the solution. Via a penalization procedure, we provide an existence and uniqueness result for these so-called constrained BSDEs with jumps. This new type of BSDE offers a nice and practical unifying framework to represent and generalize the notions of constrained BSDEs studied in [16], BSDEs with constrained jumps introduced recently by [12], as well as multidimensional BSDEs with oblique reflection presented in [11] and [9]. For example, a switching problem, represented by a multidimensional BSDE with oblique reflection, see [11], can be directly solved through a one dimensional constrained BSDE with jumps. This result is very promising from a numerical point of view for the resolution of high dimensional switching problems. All the arguments presented here rely on probabilistic tools. This allows in particular to represent non-Markovian switching problems, where, for example, the switching regime influences the dynamics of the underlined diffusion.

Key words: Switching problems, BSDE with jumps, Constrained BSDE, Reflected BSDE.

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1 Introduction

Since its introduction by Pardoux and Peng in [13], the notion of Backward Stochastic Differential Equations (BSDEs in short) has been widely extended. In particular, it appeared as a very powerful tool to solve partial differential equations (PDE) and corresponding stochastic optimization problems. Several generalizations of this notion are based on the addition of new constraints on the solution. First, El Karoui et al. [7] study the case where the component Y is forced to stay above a given process, leading to the notion of reflected BSDEs related to optimal stopping and obstacle problems. Motivated by super replication problems under portfolio constraints, Cvitanic et al. [5] consider the case where the component Z is constrained to stay in a fixed convex set. More recently, Kharroubi et al. [12] introduce a constraint on the jump component U of the BSDE, providing a representation of solutions to a class of PDE, called quasi-variational inequalities, arising in optimal impulse control problems. Generalizing the results of El Karoui et al. [7] in a multi-dimensional framework, Hu and Tang [11] followed by Hamadène and Zhang [9] consider BSDEs with oblique reflections and connect them with systems of variational inequalities and optimal switching problems. Nevertheless, they only consider cases where the switching strategy does not affect the dynamics of the underlying diffusion. Our paper introduce the notion of constrained BSDEs with jumps, which offers in particular a nice and natural probabilistic representation for these types of switching problems. This new notion essentially unifies and extends the notions of constrained BSDE without jumps, BSDE with constrained jumps as well as multidimensional BSDE with oblique reflections.

Let illustrate our presentation by the example of a switching problem and introduce an underlying diffusion process, whose dynamics are given by

$$X_t^\alpha = X_0 + \int_0^t b(X_u^\alpha, \alpha_u)du + \int_0^t \sigma(X_u^\alpha, \alpha_u)dW_u, \quad 0 \le t \le T, \tag{1.1}$$

where α is a switching control process valued in $\{1, \dots, m\}$. We consider the following switching control problem defined by

$$\sup_\alpha \mathbf{E}\Big[g(X_T^\alpha, \alpha_T) + \int_0^T f(X_s^\alpha, \alpha_s)ds + \sum_{0<\tau_k\le T} c(\alpha_{\tau_k^-}, \alpha_{\tau_k})\Big], \tag{1.2}$$

where $(\tau_k)_k$ denotes the jump times of the control α. This type of stochastic control problem is typically encountered by an agent maximizing the production rentability of a given good by switching between m possible modes of production based on different commodities. A switch is penalized by a given cost function c and, since the agent is a large actor on the market, the chosen mode of production influences the dynamics of the corresponding commodities. One of the mode of production can also be interpreted as a strategy where the agent directly buys the good on a financial market. As observed by Tang and Yong [19], the value function associated to this problem interprets on $[0, T]$ as the unique viscosity solution of a given coupled system of variational inequalities. The difficulty in the derivation of a BSDE representation for this type of problem is, firstly, the dependence of the solution in mode $i \in \{1, \dots, m\}$ with respect to the global solution in all possible modes, and

secondly, the dependence in the control of the drift and the volatility of X. As recently observed in [2], the unique solution to the corresponding system of variational inequalities interprets as the value function of a well suited stochastic target problem associated to a diffusion with jumps. Using entirely probabilistic arguments, the BSDE representation provided in this paper relies on this type of correspondence. In our approach, we let artificially the strategy jump randomly between the different modes of production. As in [14], this allows to retrieve in the jump component of a one-dimensional backward process, some information regarding the solution in the other modes of production. Indeed, let us introduce a pure jump process $(I_t)_{0\leq t\leq T}$ based on an independent random measure μ and construct the underlying process $(X_t^{I_t})_{0\leq t\leq T}$, whose dynamics are based on the random mode of production I according to equation (1.1). Let consider the following constrained BSDE associated to the two dimensional forward process (I, X^I) (called transmutation-diffusion process in [14]) and defined on $[0,T]$ by:

$$\left\{\begin{array}{lcl} Y_t & = & g(X_T^{I_T}) + \int_t^T f(X_s^{I_s}, I_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds,di)\,, \\ U_t(i) & \geq & c(i, I_{t-}), \qquad d\mathbf{P}\otimes dt\otimes\lambda(di) \;\; a.e. \end{array}\right. \tag{1.3}$$

We prove in this paper that (1.3) has one unique minimal solution which indeed relates directly to the solution of the corresponding switching problem (1.2).

In order to unify our results with the one based on BSDE with oblique reflection considered in [11] or [9], we extend this approach and introduce the notion of constrained BSDE with jumps whose solution (Y, Z, U, K) satisfies the general dynamics

$$Y_t \;\; = \;\; \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds,di), \tag{1.4}$$

a.s., for $0 \leq t \leq T$, as well as the constraint

$$h(t, Y_{t-}, Z_t, U_t(i), i) \;\; \geq \;\; 0, \qquad d\mathbf{P}\otimes dt\otimes\lambda(di) \;\; a.e.\,, \tag{1.5}$$

where f and h are given random functions. Through a penalization argument, we provide, in Section 2, existence and uniqueness of the minimal solution to this type of constrained BSDE with jumps (1.4)-(1.5). For this purpose, we present in the Appendix an extension of Peng's monotonic limit theorem [15] to the framework of BSDEs with jumps. In Section 2, we mainly extend and unify the existing literature in three directions:

- We generalize the notion of BSDE with constrained jumps considered in [12], letting the driver function f depend on U and considering general constraint function h depending on all the components of the solution.

- We add some jumps in the dynamics of constrained BSDE studied in [16] and let the coefficients depend on the jump component U.

- In the case of general non linear switching problems considered in [9], the minimal solution to our BSDE interprets nicely in terms of solution to their corresponding BSDE with oblique reflections.

The constrained BSDEs with jumps offer a natural unifying framework to represent these three distinct types of BSDE. We believe that the representation of a multidimensional obliquely reflected BSDEs by a one dimensional constrained BSDE with jumps is numerically very promising. It offers the possibility to generalize the results of [3] and develop an entirely probabilistic algorithm for solving Markovian switching problems. Furthermore, this algorithm could solve high dimensional systems of variational inequalities, which relates directly to multidimensional BSDEs with oblique reflections, see [11] for more details. The algorithm as well as the Feynman Kac representation of general constrained BSDEs with jumps is currently under study and will appear in [8].

Like all the arguments of the paper, the proof relating constrained BSDEs with jumps and BSDEs with oblique reflections only relies on probabilistic arguments and can be applied in a non-Markovian setting. In particular, we provide a new multidimensional comparison theorem, based on viability property for super-solutions to BSDE. Nevertheless, the class of reflected BSDE studied in [11] or [9] does not allow for the consideration of switching problems where the dynamics of the underlying diffusion depends in a general manner on the current switching regime. The last section of the paper deals with this type of general non Markovian switching problem, typically of the form of (1.2) where the functions g, f and c are possibly random. We relate the value process of the optimal switching problem to a well chosen family of multidimensional BSDE with oblique reflection. We finally link via a penalization procedure this family of reflected BSDEs with a member of the class of one-dimensional constrained BSDE with jumps. Therefore, constrained BSDEs with jumps offer also a nice probabilistic representation for general switching problems, even in a non-Markovian framework.

Notations. Throughout this paper we are given a finite terminal time T and a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$, and a Poisson random measure μ on $\mathbb{R}_+ \times \mathcal{I}$, where $\mathcal{I} = \{1, \ldots, m\}$, with intensity measure $\lambda(di)dt$ for some finite measure λ on $\mathcal{I}$ with $\lambda(i) > 0$ for all $i \in \mathcal{I}$. We set $\tilde{\mu}(dt, di) = \mu(dt, di) - \lambda(di)dt$ the compensated measure associated to μ. $\sigma(\mathcal{I})$ denotes the σ-algebra of subsets of $\mathcal{I}$. For $x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ with $\ell \in \mathbb{N}$, we set $|x| = \sqrt{|x_1|^2 + \cdots + |x_\ell|^2}$ the euclidean norm. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ (resp.$\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$) the augmentation of the natural filtration generated by W and μ (resp. by W), and by $\mathcal{P}$ the σ-algebra of predictable subsets of $\Omega \times [0, T]$. We denote by $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathcal{S}^{\mathbf{c},\mathbf{2}}_{\mathbb{F}}$, $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$) the set of real-valued càd-làg $\mathbb{F}$-adapted (resp. continous $\mathbb{F}$-adapted, càd-làg $\mathbb{G}$-adapted) processes $Y = (Y_t)_{0\leq t\leq T}$ such that

$$\|Y\|_{\mathcal{S}^{\mathbf{2}}} \quad := \quad \left(\mathbf{E}\Big[\sup_{0\leq t\leq T} |Y_t|^2\Big]\right)^{\frac{1}{2}} \; < \; \infty.$$

where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ (resp. $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$) denotes the completed filtration generated by W (resp. by W and μ). $\mathbf{L^p(0,T)}$, $p \geq 1$, is the set of real-valued measurable processes $\phi =$

$(\phi_t)_{0\leq t\leq T}$ such that

$$\mathbf{E}\Big[\int_0^T |\phi_t|^p dt\Big] \ < \ \infty,$$

and $\mathbf{L}^{\mathbf{p}}_{\mathbb{F}}(\mathbf{0},\mathbf{T})$ (resp. $\mathbf{L}^{\mathbf{p}}_{\mathbb{G}}(\mathbf{0},\mathbf{T})$) is the subset of $\mathbf{L}^{\mathbf{p}}(\mathbf{0},\mathbf{T})$ consisting of $\mathbb{F}$–progressively measurable (resp. $\mathbb{G}$-progressively measurable) processes.
$\mathbf{L}^{\mathbf{p}}_{\mathbb{F}}(\mathbf{W})$ (resp. $\mathbf{L}^{\mathbf{p}}_{\mathbb{G}}(\mathbf{W})$), $p\geq 1$, is the set of $\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable (resp. $\mathcal{P}$-measurable) processes $Z=(Z_t)_{0\leq t\leq T}$ such that

$$\|Z\|_{\mathbf{L}^{\mathbf{p}}(\mathbf{W})} \ := \ \Big(\mathbf{E}\Big[\int_0^T |Z_t|^p dt\Big]\Big)^{\frac{1}{p}} \ < \ \infty.$$

$\mathbf{L}^{\mathbf{p}}(\tilde{\mu})$, $p\geq 1$, is the set of $\mathcal{P}\otimes\mathcal{E}$-measurable maps $U:\Omega\times[0,T]\times\mathcal{I}\to\mathbb{R}$ such that

$$\|U\|_{\mathbf{L}^{\mathbf{p}}(\tilde{\mu})} \ := \ \Big(\mathbf{E}\Big[\int_0^T\int_{\mathcal{I}}|U_t(i)|^p\lambda(di)dt\Big]\Big)^{\frac{1}{p}} \ < \ \infty.$$

$\mathbf{A}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$) is the closed subset of $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$) consisting of nondecreasing processes $K=(K_t)_{0\leq t\leq T}$ with $K_0=0$.

For ease of notation, we omit in all the paper the dependence in $\omega\in\Omega$, whenever it is explicit.

2 Constrained Backward SDEs with jumps

This section is devoted to the presentation of constrained Backward SDEs with jumps in a framework generalizing the one considered in [12] and [16]. Namely we allow the driver function to depend on the jump component of the backward process and we extend the class of possible constraint functions by letting them depend on all the components of the solution to the BSDE. We provide here an existence and uniqueness result for this type of BSDEs and remark that they are closely related to the notion of BSDEs with oblique reflections studied by [11] and [9]. All the arguments presented here rely solely on probabilistic tools.

2.1 Formulation

A constrained BSDE with jumps is characterized by three objects :

- a terminal condition, i.e. a $\mathcal{G}_T$-measurable random variable ξ,
- a generator function, i.e. a progressively measurable map $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{\mathcal{I}}$ $\to\mathbb{R}$,
- a constraint function, i.e. a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)\otimes\sigma(\mathcal{I})$-measurable map $h:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{I}\to\mathbb{R}$ such that $h(\omega,t,y,z,.,i)$ is non-increasing for all $(\omega,t,y,z,i)\in\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{I}$.

Definition 2.1. *A solution to the corresponding constrained BSDE with jumps is a quadruple* $(Y, Z, U, K) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *satisfying*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \tag{2.1}$$

for $0 \leq t \leq T$ *a.s., as well as the constraint*

$$h(t, Y_{t^-}, Z_t, U_t(i), i) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \ \ a.e. . \tag{2.2}$$

Furthermore, (Y, Z, U, K) is referred to as the minimal solution to (2.1)-(2.2) whenever, for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ to (2.1)-(2.2), we have $Y_. \leq \tilde{Y}_.$ a.s.. In this case, Y naturally interprets in the terminology of Peng [15] as the smallest supersolution to (2.1)-(2.2).

Remark 2.1. In the case where the driver function f does not depend on U and the constraint function h is of the form $h(u + c(t, y, z), i)$, observe that this BSDE exactly fits in the framework considered in [12]. In the Brownian case (i.e. no jump component), this type of BSDEs has been studied in [16].

In order to derive an existence and uniqueness result for solutions to this type of BSDE, we require the classical Lipschitz and linear growth conditions on the coefficients as well as a constraint on the dependence of the driver function in the jump component of the BSDE. We regroup these conditions in the following assumption.

(H0)

(i) There exists a constant $k > 0$ such that the functions f and h satisfy $\mathbf{P}-$a.s. the uniform Lipschitz property

$$|f(t, y, z, (u_j)_j) - f(t, y', z', (u'_j)_j)| \leq k|(y, z, (u_j)_j) - (y', z', (u'_j)_j))|, \tag{2.3}$$

$$|h(t, y, z, u_i, i) - h(t, y', z', u'_i, i)| \leq k|(y, z, u_i) - (y', z', u'_i)| \tag{2.4}$$

for all $(t, i, y, z, (u_j)_j, y', z', (u'_j)_j) \in [0, T] \times \mathcal{I} \times [\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}]^2$.

(ii) The coefficients f and h satisfy the following growth linear condition : there exists a constant C such that $\mathbf{P}-$a.s.

$$|f(t, y, z, (u_j)_j)| + |h(t, y, z, u_i, i)| \leq C(1 + |y| + |z| + |(u_j)_j|) \tag{2.5}$$

for all $(t, i, y, z, (u_j)_j) \in [0, T] \times \mathcal{I} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}}$.

(iii) There exist two constants $C_1 \geq C_2 > -1$ such that $\mathbf{P}-$ a.s.

$$f(t, y, z, u) - f(t, y, z, u') \leq \int_{\mathcal{I}} (u_i - u'_i) \gamma_t^{y,z,u,u'}(i) \lambda(di),$$

for all $(y, z, u, u') \in \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^{\mathcal{I}}]^2$, where $\gamma^{y,z,u,u'} : \Omega \times [0, T] \times \mathcal{I} \to \mathbb{R}$ is $\mathcal{P} \otimes \sigma(\mathcal{I})-$measurable and satisfies $C_2 \leq \gamma^{y,z,u,u'} \leq C_1$.

Remark 2.2. Under Assumption **(H0)** (i) and (ii), existence and uniqueness of a solution (Y, Z, U, K) to the BSDE (2.1) with $K = 0$ follows from classical results on BSDEs with jumps, see [1] or [18] for example. In order to add the h-constraint (2.2), one needs as usual to relax the dynamics of Y by adding the non decreasing process K in (2.1). In mathematical finance, the purpose of this new process K is to increase the super replication price Y of a contingent claim, under additional portfolio constraints. In order to find a minimal solution to the constrained BSDE (2.1)-(2.2), the nondecreasing property of h is crucial for stating comparison principles needed in the penalization approach. The simpler example of constraint function to keep in mind is $h(., u, i) = c(., i) - u$, i.e. upper-bounded jumps constraint.

Remark 2.3. Part (iii) of Assumption **(H0)** constrains the form of the dependence of the driver in the jump component of the BSDE. It is inspired from [17] and will ensure comparison results for BSDEs driven by this type of driver.

2.2 Existence, uniqueness and approximation by penalization

In this paragraph, we provide an existence and uniqueness result for solutions to constrained BSDEs with jumps of the form (2.1)-(2.2). This result requires an extension of Peng's monotonic limit theorem to the case of BSDE with jump, which is presented in the Appendix.

The proof relies on a classical penalization argument and we introduce the following sequence of BSDEs with jumps

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f(s, Y_s^n, Z_s^n, U_s^n) ds + n \int_t^T \int_{\mathcal{I}} h^-(s, Y_s^n, Z_s^n, U_s^n(i), i) \lambda(di) ds \\ &\quad - \int_t^T \langle Z_s^n, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^n(i) \mu(ds, di), \quad 0 \le t \le T,\ n \in \mathbb{N}, \end{aligned} \tag{2.6}$$

where $h^-(t, y, z, u, i) := \max(-h(t, y, z, u, i), 0)$ is the negative part of the function h. Under Assumption **(H0)**, the Lipschitz property of the coefficients f and h ensures existence and uniqueness of a solution $(Y^n, Z^n, U^n) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu})$ to (2.6), see [1] or [18] . Let first state two comparison results ensuring a monotonic convergence of the sequence $(Y^n)_n$ under the additional assumption

(H1) There exists a quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ solution of (2.1)-(2.2).

This assumption may appear restrictive but is classical and we present in Section 2.3 some examples where **(H1)** is satisfied, see Remark 2.4 below for more details. Let us first state a general comparison theorem for BSDEs with jumps.

Lemma 2.1. *Let $f_1, f_2 : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathcal{I}} \to \mathbb{R}$ two generators satisfying Assumption **(H0)** and $\xi_1, \xi_2 \in \mathbf{L}^2(\Omega, \mathcal{G}_T, \mathbf{P})$. Let $(Y^1, Z^1, U^1) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ satisfying*

$$Y_t^1 = \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1, U_s^1) ds - \int_t^T \langle Z_s^1, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^1(i) \mu(ds, di) \tag{2.7}$$

and $(Y^2, Z^2, U^2, K^2) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *satisfying*

$$Y^2_t = \xi^2 + \int_t^T f_2(s, Y^2_s, Z^2_s, U^2_s)ds - \int_t^T \langle Z^2_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U^2_s(i)\mu(ds, di) + K^2_T - K^2_t \,. \tag{2.8}$$

If $\xi^1 \leq \xi^2$ *and* $f_1(t, Y^1_t, Z^1_t, U^1_t) \leq f_2(t, Y^1_t, Z^1_t, U^1_t)$ *for all* $t \in [0, T]$ *then we have* $Y^1_t \leq Y^2_t$ *for all* $t \in [0, T]$.

Proof. Let us denote $\bar{Y} := Y^2 - Y^1$, $\bar{Z} := Z^2 - Z^1$, $\bar{U} := U^2 - U^1$, $\bar{f} = f_2(., Y^2, Z^2, U^2) - f_1(., Y^1, Z^1, U^1)$ and $\bar{\xi} = \xi^2 - \xi^1$ so that

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}_s ds - \int_t^T \langle \bar{Z}_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i)\mu(ds, de) + \tilde{K}_T - \tilde{K}_t, \quad 0 \leq t \leq T, \ a.s. \tag{2.9}$$

Let now define the process a by

$$a_t = \frac{f_2(t, Y^2_t, Z^2_t, U^2_t) - f_2(t, Y^1_t, Z^2_t, U^2_t)}{\bar{Y}_t} \mathbf{1}_{\{\bar{Y}_t \neq 0\}} \,,$$

and b the $\mathbb{R}^d$-valued process defined component by component by

$$b^k_t = \frac{f_2(t, Y^1_t, Z^{(k-1)}_t, U^2_t) - f_2(t, Y^1_t, Z^{(k)}_t, U^2_t)}{V^k_t} \mathbf{1}_{\{V^k_t \neq 0\}} \,, \quad k = 1, \ldots, d \,,$$

where $Z^{(k)}_t$ is the $\mathbb{R}^d$-valued random vector whose k first components are those of Z^1 and whose $(d-k)$ lasts are those of Z^2, and V^k_t is the k-th component of $Z^{(k-1)}_t - Z^{(k)}_t$.

Notice that the processes a, b are bounded since f is Lipschitz continuous. Observe also that the process $\hat{K}$ defined by

$$\hat{K}_t = K^2_t - \int_0^t \int_{\mathcal{I}} \gamma_s^{Y^1_{s^-}, Z^1_s, U^2_s, U^1_s} \bar{U}_s(i)\lambda(di)ds + \int_0^t (f_2(s, Y^1_s, Z^1_s, U^2_s) - f_1(s, Y^1_s, Z^1_s, U^1_s))ds$$

is an increasing process according to **(H0)** (iii) and $f_1(t, Y^1_t, Z^1_t, U^1_t) \leq f_2(t, Y^1_t, Z^1_t, U^1_t)$ for all $t \in [0, T]$. With these notations, we rewrite (2.9) as:

$$\begin{aligned} \bar{Y}_t &= \bar{\xi} + \int_t^T \left(a_s \bar{Y}_s + b_s.\bar{Z}_s + \int_{\mathcal{I}} \gamma_s^{Y^1_{s^-}, Z^1_s, U^2_s, U^1_s} \bar{U}_s(i)\lambda(di) \right) ds \\ &\quad - \int_t^T \langle \bar{Z}_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i)\mu(ds, de) + \hat{K}_T - \hat{K}_t \,. \end{aligned}$$

Consider now the positive process Γ solution to the s.d.e.:

$$d\Gamma_t = \Gamma_{t^-} \left(a_t dt + \langle b_t, dW_t\rangle + \int_{\mathcal{I}} \gamma_s^{Y^1_{s^-}, Z^1_s, U^2_s, U^1_s} \mu(di, ds) \right), \quad \Gamma_0 = 1.$$

Notice that Γ lies in $\mathcal{S}^2_{\mathbb{G}}$ since a, b and γ are bounded, and Γ is positive according to **(H3)** (iii). A direct application of Itô's formula leads to

$$d[\Gamma \bar{Y}]_t = \langle \Gamma_{t^-} \bar{Z}_t + \bar{Y}_{t^-} \Gamma_{t^-} b_t, dW_t\rangle + \Gamma_{t^-} \int_{\mathcal{I}} \gamma_s^{Y^1_{s^-}, Z^1_s, U^2_s, U^1_s} \bar{U}_s(i)\tilde{\mu}(ds, di) - \Gamma_{t^-} d\hat{K}_t \,,$$

so that the process $\Gamma\bar{Y}$ is a supermartingale since $\Gamma > 0$. Hence

$$\Gamma_t \bar{Y}_t \;\; \geq \;\; \mathbb{E}\left[\Gamma_T \bar{Y}_T \middle| \mathcal{G}_t\right] \;\; = \;\; \mathbb{E}\left[\Gamma_T \bar{\xi} \middle| \mathcal{G}_t\right] \geq 0\,, \quad 0 \leq t \leq T\,,$$

leading to $\bar{Y} \geq 0$. □

We can now state our comparison results for the sequence $(Y^n)_n$.

Proposition 2.1. *Under* **(H0)***, the sequence* $(Y^n)_n$ *is nondecreasing, and, for any quadruple* $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *satisfying* (2.1)-(2.2)*, we have* $Y^n \leq \tilde{Y}$ *a.s.,* $n \in \mathbb{N}$*. Under additional Assumption* **(H1)***, the sequence of processes* (Y^n) *converges increasingly and in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0}, \mathbf{T})$ *to a process* $Y \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$*.*

Proof. The monotonic property of the sequence (Y^n) follows from a direct application of Lemma 2.1 with $f_1 = f + n\int_{\mathcal{I}} h^- d\lambda$, $f_2 = f + (n+1)\int_{\mathcal{I}} h^- d\lambda$ and $K^2 \equiv 0$. For any quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying (2.1)-(2.2), we obtain $Y^n \leq \tilde{Y}$ a.s., $n \in \mathbb{N}$, taking $f_1 = f_2 = f + n\int_{\mathcal{I}} h^- d\lambda$ in the previous lemma. Under additional Assumption **(H1)**, the sequence (Y^n) is therefore increasing and upper bounded, ensuring its monotonic and in $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0}, \mathbf{T})$ convergence. □

We now turn to the convergence of the triplet $(Z^n, U^n, K^n)_n$ where, for any $n \in \mathbb{N}$, the increasing process $K^n \in \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ is defined by

$$K^n_t \;\; = \;\; n\int_0^t \int_{\mathcal{I}} h^-(s, Y^n_s, Z^n_s, U^n_s(i), i)\lambda(di)ds, \quad 0 \leq t \leq T.$$

For this purpose, we provide in Theorem 4.3 of the appendix an extension of Peng's monotonic limit theorem [15] to BSDEs with jumps.

Theorem 2.1. *Under* **(H0)**-**(H1)***, there exists a unique minimal solution* $(Y, Z, U, K) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *to* (2.1)-(2.2)*, with* K *predictable. Furthermore* Y *is the increasing limit of* $(Y^n)_n$*,* K *is the weak limit of* $(K^n)_n$ *in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0}, \mathbf{T})$*, and*

$$\|Z^n - Z\|_{\mathbf{L^p(W)}} + \|U^n - U\|_{\mathbf{L^p}(\tilde{\mu})} \;\; \longrightarrow \;\; 0,$$

for any $p \in [1, 2)$*, as* n *goes to infinity.*

Proof. From Proposition 2.1 and Theorem 4.3, we derive the convergence of the sequence $(Y^n, Z^n, U^n, K^n)_n$ to (Y, Z, U, K), solution to (2.1). The constraint (2.2) is satisfied by (Y, Z, U, K) since the sequence $(K^n)_n$ is bounded in $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$, see (4.24) in the proof of Theorem 4.3. Observe also that K inherits the predictability of K^n, $n \in \mathbb{N}$. The uniqueness of the minimal solution (Y, Z, U, K) follows from the identification of the predictable, continuous and bounded variation parts. □

Remark 2.4. Observe that the purpose of Assumption **(H1)** is to ensure an upper bound on the sequence $(Y^n)_n$ of solutions to the penalized BSDEs. If such an upper bound already exists, this assumption is not required anymore but will be automatically satisfied from the existence of a minimal solution to the constrained BSDE with jumps. Cases where Assumption **(H1)** is satisfied are presented in the next section, and sufficient conditions for this assumption in a markovian setting are presented in [8].

Remark 2.5. Notice that the convergence of Y^n to Y, which is obtained in a weak sense (i.e. in $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0},\mathbf{T})$) could be improved in the Markovian case (i.e. $Y^n \to Y$ in $\mathcal{S}^2_{\mathbb{G}}$). In this case we get the convergence (Z^n, U^n) to (Z, U) in $\mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu})$. See [8] for more details.

2.3 Link with multi-dimensional reflected Backward SDEs

In this section, we prove that multidimensional reflected BSDEs introduced by [11] and generalized by [9] are closely related to constrained BSDEs with jumps. The arguments presented here are purely probabilistic and therefore apply in the non Markovian framework considered in [11]. Furthermore, the proofs require precise comparison results based on viability properties that are reported in the Appendix for the convenience of the reader.

Recall that solving a multidimensional reflected BSDE consists in finding m triplets $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in (\mathcal{S}^{\mathbf{c},\mathbf{2}}_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^{\mathcal{I}}$ satisfying

$$\begin{cases} Y^i_t = \xi^i + \int_t^T \psi_i(s, Y^1_s, \ldots, Y^m_s, Z^i_s)ds - \int_t^T \langle Z^i_s, dW_s\rangle + K^i_T - K^i_t \\ Y^i_t \geq \max_{j\in A_i} h_{i,j}(t, Y^j_t) \\ \int_0^T [Y^i_t - \max_{j\in A_i}\{h_{i,j}(t, Y^j_t)\}]dK^i_t = 0 \end{cases} \tag{2.10}$$

where, for all $i \in \mathcal{I}$, $\psi_i : \ \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is an $\mathbb{F}$-progressively measurable map, $\xi^i \in \mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbf{P})$, A_i is a nonempty subset of $\mathcal{I} \setminus \{i\}$, and , for any $j \in A_i$, $h_{i,j} : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given function. As detailed in [11], existence and uniqueness of a solution to (2.10) is ensured by the following assumption.

(H2)

(i) For any $i \in \mathcal{I}$ and $j \in A_i$, we have $\xi^i \geq h_{i,j}(T, \xi^j)$.

(ii) For any $i \in \mathcal{I}$, $\mathbf{E}\int_0^T \sup_{y\in\mathbb{R}^m|y_i=0} |\psi_i(t, y, 0)|^2 dt + \mathbf{E}|\xi^i|^2 < +\infty$, and ψ_i is Lipschitz continuous: there exists a constant $k_\psi \geq 0$ such that

$$|\psi_i(t, y, z) - \psi_i(t, y', z')| \leq k_\psi(|y - y'| + |z - z'|) , \quad \forall (i, y, z, y', z') \in \mathcal{I} \times [\mathbb{R} \times \mathbb{R}^d]^2 .$$

(iii) For any $i \in \mathcal{I}$, and $j \neq i$, ψ_i is increasing in its $(j + 1)-$th variable i.e. for any $(t, y, y', z) \in \mathcal{I} \times [\mathbb{R}^m]^2 \times \mathbb{R}^d$ such that $y_k = y'_k$ for $k \neq j$ and $y_j \leq y'_j$ we have

$$\psi_i(t, y, z) \quad \leq \quad \psi_i(t, y', z) \quad \mathbf{P} - a.s.$$

(iv) For any $(i, t, y) \in \mathcal{I} \times [0, T] \times \mathbb{R}$ and $j \in A_i$, $h_{i,j}$ is continuous, $h_{i,j}(t, .)$ is a 1-Lipschitz and increasing function satisfying $h_{i,j}(t, y) \leq y$. Furthermore, for any $l \in A_j$, we have $l \in A_i \cup \{i\}$ and $h_{i,l}(t, y) > h_{i,j}(t, h_{j,l}(t, y))$.

Remark 2.6. Part (ii) and (iii) of Assumption **(H2)** are classical Lipschitz and monotony properties of the driver. Part (iv) ensures a tractable form for the domain of $\mathbb{R}^m$ where $(Y^i)_{i\in\mathcal{I}}$ lies, and (i) implies that the terminal condition is indeed in the domain,

Consider now the following constrained BSDE with jump: find a minimal quadruple $(\tilde Y,\tilde Z,\tilde U,\tilde K)\in\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}\times\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W})\times\mathbf{L}^{\mathbf{2}}(\tilde\mu)\times\mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying

$$\begin{aligned}\tilde Y_t &= \xi^{I_T}+\int_t^T\psi_{I_s}(s,\tilde Y_s+\tilde U_s(1)\mathbf{1}_{I_s\neq1},\ldots,\tilde Y_s+\tilde U_s(m)\mathbf{1}_{I_s\neq m},\tilde Z_s)ds+\tilde K_T-\tilde K_t \qquad (2.11)\\ &\quad -\int_t^T\langle\tilde Z_s,dW_s\rangle-\int_t^T\int_{\mathcal{I}}\tilde U_s(i)\mu(ds,di),\quad 0\le t\le T,\ a.s.\end{aligned}$$

with

$$\mathbf{1}_{A_{I_{t^-}}}(i)\Big[\tilde Y_{t^-}-h_{I_{t^-},i}(t,\tilde Y_{t^-}+\tilde U_t(i))\Big]\ \ge\ 0,\qquad d\mathbf{P}\otimes dt\otimes\lambda(di)\ \ a.e. \qquad (2.12)$$

where the process I is a pure jumps process defined by

$$I_t\ =\ I_0+\int_0^t\int_{\mathcal{I}}(i-I_{s^-})\mu(ds,di)\,.$$

Remark that, if $\mu=\sum_{n\ge0}\delta_{(\tau_n,A_n)}$, the process I is simply the pure jump process which coincides with A_n on each $[\tau_n,\tau_{n+1})$. This BSDE enters obviously into the class of constrained BSDEs with jumps of the form (2.1)-(2.2) studied above, with the following correspondence

$$\xi=\xi^{I_T}\,,\ \ f(t,y,z,u)=\psi_{I_t}(t,(y+u_i\mathbf{1}_{I_t\neq i})_{i\in\mathcal{I}},z)\ \ \text{and}\ \ h(t,y,z,v,i)=y-h_{I_{t^-},i}(t,y+v)\,.$$

Observe further that Assumption **(H0)** is automatically satisfied under **(H2)**, and, as detailed in the next proposition, **(H2)** also implies **(H1)** for (2.11)-(2.12) and its minimal solution can be directly related to the solution of the BSDE with oblique reflections (2.10).

Proposition 2.2. *Let Assumption* **(H2)** *hold and* $((Y^1,Z^1,K^1),\ldots,(Y^m,Z^m,K^m))$ *be the solution to* (2.10). *Then* **(H1)** *holds true for* (2.11)-(2.12) *and, if we denote* $(\tilde Y,\tilde Z,\tilde U,\tilde K)$ *the minimal solution to* (2.11)-(2.12), *the following equality holds*

$$\tilde Y_t=Y_t^{I_t},\quad \tilde Z_t=Z_t^{I_{t^-}}\quad\text{and}\qquad U_t(.)=Y_t^{i}-Y_{t^-}^{I_t}.$$

In order to derive this result, we need to introduce and discuss the corresponding penalized BSDEs. For $n\in\mathbb{N}$, consider the system of penalized BSDEs: find m couples $(Y^{i,n},Z^{i,n})_{i\in\mathcal{I}}\in(\mathcal{S}^{\mathbf{c},\mathbf{2}}_{\mathbb{F}}\times\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^{\mathcal{I}}$ satisfying

$$\begin{aligned}Y_t^{i,n} &= \xi^i+\int_t^T\psi_i(s,Y_s^{1,n},\ldots,Y_s^{m,n},Z_s^{i,n})ds-\int_t^T\langle Z_s^{i,n},dW_s\rangle\\ &\quad +n\int_t^T\sum_{j\in A_i}[Y_s^{i,n}-h_{i,j}(s,Y_s^{j,n})]^-\lambda(j)ds,\quad 0\le t\le T,\ a.s. \qquad (2.13)\end{aligned}$$

Lemma 2.2. *Under* **(H2)**, *the sequence* $(Y^{i,n})_n$ *is increasing and converges to* Y^i $d\mathbf{P}\otimes dt$ *a.e. and in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{0},\mathbf{T})$, *and the sequence* $(Z^{i,n})_n$ *converges weakly to* Z^i *in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W})$, *for all* $i\in\mathcal{I}$.

Proof. For any $t\in[0,T]$, $y\in\mathbb{R}^m$ and $z\in[\mathbb{R}^d]^m$, set

$$\psi_i^n(t,y,z)\ \ :=\ \ \psi_i(t,y,z_i)+n\sum_{j\in A_i}[y_i-h_{i,j}(t,y_j)]^-\lambda(j).$$

From **(H2)** (iii) and (iv), we have $\psi_i^n(t, y+y', z) \geq \psi_i^n(t, y, z)$ for any $y' \in (\mathbb{R}^+)^m$ such that $y'_i = 0$. Since ψ_i^n depends only on z_i, it satisfies inequality (4.23) in the Appendix. Therefore, Theorem 4.2 leads to

$$Y_t^i \geq Y_t^{i,n+1} \geq Y_t^{i,n} \quad \text{for all} \quad (t,i,n) \in [0,T] \times \mathcal{I} \times \mathbb{N}\,. \tag{2.14}$$

By Peng's monotonic limit theorem, there exist m càdlàg processes $\hat{Y}^1, \ldots, \hat{Y}^m \in \mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$, $\hat{Z}^1, \ldots, \hat{Z}^m \in \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W})$ and $\hat{K}^1, \ldots, \hat{K}^m \in \mathbf{A}^{\mathbf{2}}_{\mathbb{F}}$, such that $Y^{i,n} \uparrow \hat{Y}^i$ a.e., $Y^{i,n} \to \hat{Y}^i$ in $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$, $Z^{i,n} \to \hat{Z}^i$ in $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$ weakly and

$$\hat{Y}_t^i \quad = \quad \xi^i + \int_t^T \psi_i(s, \hat{Y}_s^1, \ldots, \hat{Y}_s^m, \hat{Z}_s^i)ds - \int_t^T \langle \hat{Z}_s^i, dW_s \rangle + \hat{K}_T^i - \hat{K}_t^i\,,$$

with $Y_t^i \geq \max_{j \in A_i} h_{i,j}(t, Y_t^j)$. Then, using the same arguments as in the proof of Theorem 2.4 in [9], we prove that $(\hat{Y}, \hat{Z}, \hat{K})$ is the unique solution to (2.10). □

Proof of Proposition 2.2. For $i \in \mathcal{I}$ and $n \in \mathbb{N}$, define the processes $Y^{I,n} \in \mathcal{S}^2_{\mathbb{G}}$, $Z^{I,n} \in \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W})$ and $U^{I,n} \in \mathbf{L}^{\mathbf{2}}(\tilde{\mu})$ by

$$Y_t^{I,n} \;:=\; Y_t^{I_t,n}\,, \quad Z_t^{I,n} \;:=\; Z_t^{I_{t^-},n} \quad \text{and} \quad U_s^{I,n}(i) \;:=\; Y_s^{i,n} - Y_{s^-}^{I,n}\,, \quad 0 \leq t \leq T. \tag{2.15}$$

We deduce from (2.13) that $(Y^{I,n}, Z^{I,n}, U^{I,n})$ is the solution to the penalized BSDE associated to (2.11)-(2.12) given by

$$\begin{aligned} Y_t^{I,n} \quad &= \quad \xi^{I_T} + \int_t^T \psi_{I_s}(s, Y_{s^-}^{I,n} + U_s^{I,n}(1)\mathbf{1}_{I_s \neq 1}, \ldots, Y_{s^-}^{I,n} + U_s^{I,n}(m)\mathbf{1}_{I_s \neq m}, Z_s^{I,n})ds \\ &- \quad \int_t^T \langle Z_s^{I,n}, dW_s \rangle + n \int_t^T \int_{\mathcal{I}} h^-(s, Y_{s^-}^{I,n}, Z_s^{I,n}, U_s^{I,n}(i))\lambda(i)ds + \int_t^T \int_{\mathcal{I}} U_s^{I,n}(i)\mu(di, ds)\,. \end{aligned}$$

From (2.14), the sequence $(Y^{I,n})_n$ is bounded in $\mathcal{S}^2_{\mathbb{G}}$ and, proceeding as in Section 2, we prove that

$$\|Y^{I,n} - \tilde{Y}\|_{\mathbf{L^2(0,T)}} + \|Z^{I,n} - \tilde{Z}\|_{\mathbf{LP(W)}} + \|U^{I,n} - \tilde{U}\|_{\mathbf{LP}(\tilde{\mu})} \quad \longrightarrow \quad 0\,,$$

where $(\tilde{Y}, \tilde{Z}, \tilde{U})$ is the minimal solution to (2.11)-(2.12). Combined with (2.15) and Lemma 2.2, this concludes the proof. □

The main interest of the previous result is the unification of the notion of constrained BSDEs without jumps studied in [16], the class of BSDEs with constrained jumps introduced by [12] and the notion of BSDEs with oblique reflections considered in [9] and [11]. As a by product, we deduce from [11] that constrained BSDE with jump of the form interpret as a viscosity solution to a system of variational inequalities. This opens the door to the numerical approximation of solution to this type of PDEs by purely probabilistic schemes in the spirit of the algorithm presented in [3]. Instead of considering a multidimensional BSDE with oblique reflections, one just needs to approximate a one dimensional constrained BSDE with jumps. The Feynman-Kac representation of general constrained BSDE with jumps and the corresponding numerical algorithm are under study and will

appear in a separate paper [8].

Since the results presented here rely on probabilistic arguments, they can apply to eventually non Markovian settings considered in [9] or [11]. Nevertheless, they do not include cases where the dynamics of the underlying diffusion depends on the value of the current switching regime. The purpose of the next section is to extend the link presented here to a more general class of non-Markovian switching problems.

3 Constrained Backward SDEs with jumps and non-Markovian switching

This section is devoted to the interpretation of non-Markovian switching problems in terms of solutions to BSDEs with constrained jumps. In particular, we consider useful cases where the current switching regime influences the dynamics of the underlying diffusion. This is the case for example if we consider a producer who is a large investor on the commodity market who influences the dynamics of the underlying commodity prices. One of the dimension of the underlying can also be the level of the stock in some commodity which is of course directly related to the chosen mode of production. To our knowledge, no BSDE representation has yet been established in this type of framework. We first extend the results of [11] and relate the solution to a general non-Markovian switching problem with a well chosen family of multidimensional BSDE with oblique reflections. We finally link this family of BSDE with one single one-dimensional constrained BSDE with jumps leading to the announced representation property.

3.1 Non-Markovian optimal switching

Given the set $\mathcal{I} = \{1, \dots, m\}$ and a terminal time $T < +\infty$, an impulse strategy α consists in a sequence $\alpha := (\tau_k, \zeta_k)_{k \geq 1}$, where $(\tau_k)_{k \geq 1}$ is an increasing sequence of $\mathbb{F}$-stoppping times, and ζ_i are $\mathcal{F}_{\tau_i}$-measurable random variables valued in $\mathcal{I}$. To a strategy $\alpha = (\tau_k, \zeta_k)_{k \geq 1}$ and an initial regime i_0, we naturally associate the process $(\alpha_t)_{t \leq T}$ defined by

$$\alpha_t \quad := \quad \sum_{k \geq 0} \zeta_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t)\,.$$

with $\tau_0 = 0$ and $\zeta_0 = i_0$. We denote by $\mathcal{A}$ the set of admissible strategies. Given a strategy $\alpha \in \mathcal{A}$ and an initial condition (i_0, X_0), we define the controlled process X^α by

$$X^\alpha_t = X_0 + \int_0^t b(s, \alpha_s, X^\alpha_s) ds + \int_0^t \sigma(s, \alpha_s, X^\alpha_s) dW_s\,, \tag{3.1}$$

and we consider the total profit at horizon T defined by

$$J(\alpha) \quad := \quad \mathbf{E}\left[g(\alpha_T, X^\alpha_T) + \int_0^T \psi(s, \alpha_s, X^\alpha_s) ds + \sum_{0 < \tau_k \leq T} c(\tau_k, \zeta_{k-1}, \zeta_k) \right]\,. \tag{3.2}$$

We suppose here that the functions $b, \sigma, \psi : \Omega \times [0,T] \times \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d\times d}, \mathbb{R}$ are progressively measurable functions and $g : \Omega \times \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{F}_T \otimes \sigma(\mathcal{I}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and that $c : \Omega \times [0,T] \times \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ is a progresively measurable function.

Given an initial data (X_0, i_0), the switching problem consists in finding a strategy $\alpha^* \in \mathcal{A}$ such that

$$J(\alpha^*) = \sup_{\alpha\in\mathcal{A}} J(\alpha)$$

Such a strategy is called optimal and we shall work under the following assumption.

(H3)

(i) b and σ satisfy the Lipschitz property: there exists a constant k such that $\mathbf{P}-$a.s.

$$|b(\omega,t,i,x) - b(\omega,t,i,x')| + |\sigma(\omega,t,i,x) - \sigma(\omega,t,i,x')| \quad \leq \quad k\,|x - x'|\,,$$

for all $(\omega,t,i,x,x') \in \Omega \times [0,T] \times \mathcal{I} \times \mathbb{R}^d \times \mathbb{R}^d$.

(ii) The terminal condition g satisfies the following structural condition

$$g(\omega,x,i) \geq \max_{j\in\mathcal{I}}\{g(\omega,x,j) + C(\omega,T,i,j)\} \quad \forall(\omega,i,x) \in \Omega \times \mathbb{R}^d \times \mathcal{I},$$

(iii) The functions g and ψ are bounded, i.e. there exists two constants $\bar{g}$ and $\bar{\psi}$ satisfying

$$\sup_{(\omega,i,x)\in\Omega\times\mathcal{I}\times\mathbb{R}^d}\{|g(\omega,i,x)|\} \leq \bar{g} \quad \text{and} \quad \sup_{(\omega,t,i,x)\in\Omega\times[0,T]\times\mathcal{I}\times\mathbb{R}^d}\{|\psi(\omega,t,i,x)|\} \leq \bar{\psi}\,.$$

(iv) The cost function c is upper-bounded, i.e. there exists a constant $\bar{c} > 0$ such that

$$\max_{(\omega,t,i,j)\in\Omega\times[0,T]\times\mathcal{I}\times\mathcal{I}} c(\omega,t,i,j) \leq -\bar{c}\,.$$

Furthermore $c(.,i,j)$ is continuous, for all $i, j \in \mathcal{I}$, and it satisfies the structural condition

$$c(\omega,t,i,l) > c(\omega,t,i,j) + c(\omega,t,j,l)\,, \quad \forall(\omega,t,i,j,l) \in \Omega\times[0,T]\times[\mathcal{I}]^3 \text{ s.t. } j \neq i\,,\ j \neq l\,.$$

Remark 3.1. Part (i) of Assumption **(H3)** provides existence and uniqueness of a solution to (3.1). Part (ii) ensures the non-optimality of a switching at maturity, (iv) makes indirect switching strategy irrelevant and (iii)-(iv) ensures the problem is well posed.

Let define the set of finite strategies $\mathcal{D}$ by

$$\mathcal{D} \quad := \quad \{\alpha = (\tau_k, \zeta_k)_{k\geq 1} \in \mathcal{A} \mid \mathbf{P}(\tau_k < T, \ \ \forall k \geq 1) = 0\}$$

Let first observe the following property:

Proposition 3.1. *Under **(H3)**, the supremum of J over $\mathcal{A}_{0,i}$ coincides with the one over $\mathcal{D}_{0,i}$, that is*

$$\sup_{\alpha\in\mathcal{A}} J(\alpha) = \sup_{\alpha\in\mathcal{D}} J(\alpha)\,, \qquad \forall i \in \mathcal{I}\,. \tag{3.3}$$

Proof. Fix $i \in \mathcal{I}$. Consider a strategy $\alpha = (\tau_k, \zeta_k)_{k\geq 0} \in \mathcal{A} \setminus \mathcal{D}$ and define $B := \{\omega \in \Omega \mid \tau_n(\omega) < T,\ \forall n \in \mathbb{N}^*\}$ so that $\mathbf{P}(B) > 0$. Such a strategy is not optimal since we derive from **(H3)** (iii) and (iv) that

$$J(\alpha) \quad \leq \quad \bar{g} + T\bar{\psi} + \mathbf{E}\left[\sum_{0<\tau_k\leq T} c(\tau_k, \zeta_{k-1}, \zeta_k)\mathbf{1}_B\right] + \mathbf{E}\left[\sum_{0<\tau_k\leq T} c(\tau_k, \zeta_{k-1}, \zeta_k)\mathbf{1}_{B^c}\right] \;=\; -\infty\,.$$

□

3.2 Reflected BSDEs and optimal switching

Following the approach of [6], we consider in this section a family of reflected BSDE. For any couple (ν, η) with ν a stopping time valued in $[0, T]$ and η a $\mathcal{F}_\nu$-measurable random variable taking values in $\mathbb{R}^d$, we consider the following reflected BSDE

$$\begin{cases} (Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}} \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^{\mathcal{I}}, \\ Y^{\nu,i,\eta}_t = g(i, X^{\nu,i,\eta}_T) + \int_t^T \psi(s, i, X^{\nu,i,\eta}_s)\mathbf{1}_{s\geq\nu} ds - \int_t^T \langle Z^{\nu,i,\eta}_s, dW_s\rangle + K^{\nu,i,\eta}_T - K^{\nu,i,\eta}_t, \\ Y^{\nu,i,\eta}_t \geq \max_{j\in\mathcal{I}}\{Y^{\nu,j,\eta}_t + c(t, i, j)\}, \\ \int_0^T [Y^{\nu,i,\eta}_t - \max_{j\in\mathcal{I}}\{Y^{\nu,j,\eta}_t + c(t, i, j)\}] dK^{\nu,i,\eta}_t \;=\; 0, \end{cases} \tag{3.4}$$

where $X^{\nu,i,\eta}$ is the diffusion defined by

$$X^{\nu,i,\eta}_t \;=\; \eta\mathbf{1}_{t\geq\nu} + \int_0^t b(s, i, X^{\nu,i,\eta}_s)\mathbf{1}_{s\geq\nu} ds + \int_0^t \sigma(s, i, X^{\nu,i,\eta}_s)\mathbf{1}_{s\geq\nu} dW_s\,, \qquad \forall t \geq 0. \tag{3.5}$$

Under **(H3)**, we know from [9] that (3.4) has a unique solution for any stopping time ν and any $\mathcal{F}_\nu$-measurable random variable η, and we denote by $\mathcal{O}^{\nu,\cdot,\eta}$ its barrier defined by

$$O^{\nu,i,\eta}_t \;:=\; \max_{j\in\mathcal{I}}\{Y^{\nu,j,\eta}_t + c(t, i, j)\}\,, \qquad i \in \mathcal{I}\,,\ \ t \leq T. \tag{3.6}$$

We aim at relating the solutions to this class of reflected BSDEs to the solution of the optimal non Markovian switching problem presented in (3.2). The next proposition relates a stability property, a Snell envelope representation and a global estimate on the family of processes $(Y^{\nu,\cdot,\eta})_{(\nu,\eta)}$.

Proposition 3.2. *Assume that **(H3)** holds and take ν,ν' two $\mathbb{F}$-stopping times such that $\nu \leq \nu'$ and η an $\mathcal{F}_\nu$-measurable random variable valued in $\mathbb{R}^d$.*

(i) For all $i \in \mathcal{I}$ and $t \geq \nu'$, we have $Y^{\nu,i,\eta}_t = Y^{\nu',i,X^{\nu,i,\eta}_{\nu'}}_t$, $\mathbf{P}$-a.s.

(ii) For all $i \in \mathcal{I}$ and $t \geq \nu'$, we have the following representation

$$Y^{\nu,i,\eta}_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbf{E}\left[\int_t^\tau \psi(s, i, X^{\nu,i,\eta}_s)\mathbf{1}_{s\geq\nu} ds + \mathcal{O}^{\nu,i,\eta}_\tau \mathbf{1}_{\tau<T} + g(i, X^{\nu,i,\eta}_T)\mathbf{1}_{\tau=T}\,\middle|\, \mathcal{F}_t\right]. \tag{3.7}$$

(iii) There exists a constant $\bar{Y}$ such that

$$\sup_{(t,\nu,i,\eta)} |Y_t^{\nu,i,\eta}| \leq \bar{Y}\,. \tag{3.8}$$

Proof. (i) Notice first that $X^{\nu,i,\eta}$ and $X^{\nu',i,X_{\nu'}^{\nu,\zeta,\eta}}$ solve the same SDE, namely

$$X_{\nu'} = X_{\nu'}^{\nu,i,\eta} \quad \text{and} \quad dX_t = b(t,i,X_t)dt + \sigma(t,i,X_t)dW_t \quad \text{for } t \geq \nu'. \tag{3.9}$$

Under **(H3)** (i), equation (3.9) admits a unique solution and we have $X^{\nu',i,X_{\nu'}^{\nu,i,\eta}} = X^{\nu,i,\eta}$ on $[\nu',T]$. We deduce that $(Y^{\nu',i,X_{\nu'}^{\nu,i,\eta}}, Z^{\nu',i,X_{\nu'}^{\nu,i,\eta}}, K^{\nu',i,X_{\nu'}^{\nu,i,\eta}}))_{i\in\mathcal{I}}$ satisfies the same BSDE as $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}}$ on $[\nu',T]$. Uniqueness of solution to this BSDE is given by [9].

(ii) Regarding of (3.4), $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})$ interprets as the solution to a reflected BSDE with single barrier $\mathcal{O}^{\nu,i,\eta}$. We deduce from [7] that $Y^{\nu,i,\eta}$ admits the snell envelope representation (3.7).

(iii) For fixed ν and η, the family $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}}$ is the solution to the BSDE with oblique reflection (3.4). We know from [9] that $(Y^{\nu,i,\eta,n}, Z^{\nu,i,\eta,n}, K^{\nu,i,\eta,n})$ converges to $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})$, where the sequence $(Y^{\nu,i,\eta,n}, Z^{\nu,i,\eta,n}, K^{\nu,i,\eta,n})_n$ is defined recursively by

$$Y_t^{\nu,i,\eta,0} \;=\; g(i,X_T^{\nu,i,\eta}) + \int_t^T \psi(s,i,X_T^{\nu,i,\eta})\mathbf{1}_{s\geq\nu}ds - \int_t^T \langle Z_s^{\nu,i,\eta,0}, dW_s\rangle \quad \text{and} \quad K_t^{\nu,i,\eta,0} \;=\; 0,$$

and, for $n \geq 1$,

$$\left\{\begin{array}{l} Y_t^{\nu,i,\eta,n} = g(i,X_T^{\nu,i,\eta}) + \int_t^T \psi(s,i,X_s^{\nu,i,\zeta})\mathbf{1}_{s\geq\nu}ds - \int_t^T \langle Z_t^{\nu,i,\eta,n}, dW_s\rangle + K_T^{\nu,i,\eta,n} - K_t^{\nu,i,\eta,n},\\ Y_t^{\nu,i,\eta,n} \geq \max_{j\in\mathcal{I}}\{Y_t^{\nu,j,\eta,n-1} + c(t,i,j)\},\\ \int_0^T [Y_t^{\nu,i,\eta,n} - \max_{j\in\mathcal{I}}\{Y_t^{\nu,j,\eta,n-1} + c(t,i,j)\}]dK_t^{\nu,i,\eta,n} \;=\; 0\,. \end{array}\right. \tag{3.10}$$

To derive (3.8), it suffices to prove by induction on n that

$$|Y_t^{\nu,i,\eta,n}| \leq (T-t+1)\max\{\bar{\psi},\bar{g}\}\,, \quad i\in\mathcal{I}\,,\ \ 0\leq t\leq T\,,\ \ n\in\mathbb{N}\,.$$

First, rewriting $Y^{\nu,i,\eta,0}$ as a conditional expectation, we derive

$$|Y_t^{\nu,i,\eta,0}| \leq (T-t)\bar{\psi} + \bar{g} \leq (T-t+1)\max\{\bar{\psi},\bar{g}\}\,, \qquad 0\leq t\leq T\,,\ \ i\in\mathcal{I}\,.$$

Fix $n\in\mathbb{N}$ and suppose the result is true for $Y^{\cdot,\cdot,n}$. Using the representation of $Y^{\nu,i,\eta,n+1}$ as a Snell envelope, we derive

$$Y_t^{\nu,i,\eta,n+1} \;=\; \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbf{E}\left[\int_t^\tau \psi(s,i,X_s^{\nu,i,\eta})\mathbf{1}_{s\geq\nu}ds + \mathcal{O}_\tau^{\nu,i,\eta,n+1}\mathbf{1}_{\tau<T} + g(i,X_T^{\nu,i,\eta})\mathbf{1}_{\tau=T}\,\middle|\,\mathcal{F}_t\right]\,,$$

where $\mathcal{O}_\tau^{\nu,i,\eta,n+1} := \max_{j\in\mathcal{I}}\{Y_\tau^{\nu,j,\eta,n} + c(t,i,j)\}$. Combining this representation with Assumption **(H3)** leads to $|Y_t^{\nu,i,\eta,n+1}| \leq (T-t+1)\max\{\bar{g},\bar{\psi}\}$ and concludes the proof. $\square$

For any stopping time ν, any $\mathcal{F}_\nu$-random variable η and any $\mathcal{I}$-valued random variable ζ, we naturally introduce the processes $Y^{\nu,\zeta,\eta}$ and $\mathcal{O}^{\nu,\zeta,\eta}$ by

$$Y_t^{\nu,\zeta,\eta} = \sum_{i\in\mathcal{I}} Y_t^{\nu,i,\eta}\mathbf{1}_{\zeta=i} \quad \text{and} \quad \mathcal{O}_t^{\nu,\zeta,\eta} = \sum_{i\in\mathcal{I}} \mathcal{O}_t^{\nu,i,\eta}\mathbf{1}_{\zeta=i}. \tag{3.11}$$

We are now able to state the main results of this section characterizing the optimal solution to the switching problem (3.2) in terms of reflected BSDEs.

Theorem 3.1. *Let $\alpha^* = (\tau_n^*, \zeta_n^*)_{n\geq 0}$ be the strategy given by $\alpha_0^* = (0, i_0)$ and defined recursively for $n \geq 1$ by*

$$\tau_n^* := \inf\left\{ s \geq \tau_{n-1}^* \ ; \ Y_s^{\tau_{n-1}^*,\zeta_{n-1}^*,X^*_{\tau_{n-1}^*}} = \mathcal{O}_s^{\tau_{n-1}^*,\zeta_{n-1}^*,X^*_{\tau_{n-1}^*}} \right\}, \tag{3.12}$$

$$\zeta_n^* \quad \text{is} \quad \text{s.t.} \quad \mathcal{O}_{\tau_n^*}^{\tau_{n-1}^*,\zeta_{n-1}^*,X^*_{\tau_{n-1}^*}} = Y_{\tau_n^*}^{\tau_n^*,\zeta_n^*,X^*_{\tau_n^*}} + c(\tau_n^*, \zeta_{n-1}^*, \zeta_n^*), \tag{3.13}$$

with X^* the diffusion defined by

$$X_t^* = x_0 + \sum_{n\geq 1} \int_{\tau_{n-1}^*}^{\tau_n^*} b(s, \zeta_{n-1}^*, X_s^*)\mathbf{1}_{s\leq t} ds + \int_{\tau_{n-1}^*}^{\tau_n^*} \sigma(s, \zeta_{n-1}^*, X_s^*)\mathbf{1}_{s\leq t} ds, \quad t \geq 0. \tag{3.14}$$

*Under Assumption **(H3)**, the strategy α^* is optimal for the switching problem* (3.2) *and we have*

$$Y_0^{i_0}(0, x_0) = J(\alpha^*). \tag{3.15}$$

Proof. The proof is performed in two steps.

Step 1. *The strategy $\alpha^* \in \mathcal{D}$ and satisfies $Y_0^{0,i_0,x_0} = J(\alpha^*)$.*
The representation (3.7) rewrites

$$Y_0^{0,i_0,x_0} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbf{E}\left[\int_0^\tau \psi(s, i_0, X_s^{0,i_0,x_0}) ds + \mathcal{O}_\tau^{0,i_0,x_0}\mathbf{1}_{\tau<T} + g(i_0, X_T^{0,i_0,x_0})\mathbf{1}_{\tau=T} \right]. \tag{3.16}$$

Since the boundary $\mathcal{O}^{0,i_0,x_0}$ is continuous, the stopping time τ_1^* is optimal for (3.16) and we get

$$Y_0^{0,i_0,x_0} = \mathbf{E}\left[\int_0^{\tau_1^*} \psi(s, i_0, X_s^{0,i_0,x_0}) ds + \mathcal{O}_{\tau_1^*}^{0,i_0,x_0}\mathbf{1}_{\tau_1^*<T} + g(i_0, X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1^*=T} \right].$$

If $\tau_1^* = T$, the proof is over. Let suppose that $\tau_1^* < T$ and, according to the definition of ζ_1^*, we derive

$$Y_0^{0,i_0,x_0} = \mathbf{E}\left[\int_0^{\tau_1^*} \psi(s, i_0, X_s^{0,i_0,x_0}) ds + Y_{\tau_1^*}^{\tau_1^*,\zeta_1^*,X^{0,i_0,x_0}_{\tau_1^*}} + c(\tau_1^*, i_0, \zeta_1^*) \right].$$

Similarly, we can use the representation of $Y_{\tau_1^*}^{\tau_1^*,\zeta_1^*,X_{\tau_1^*}^{0,i_0,x_0}}$ given by (3.7), and we deduce recursively that

$$Y_0^{0,i_0,x_0} = \mathbf{E}\left[\int_0^{\tau_n^*}\psi(s,i_0,X_s^*)ds + Y_{\tau_n^*}^{\tau_n^*,\zeta_n^*,X_{\tau_n^*}^*} + \sum_{0<k\leq n} c(\tau_k,\zeta_{k-1}^*,\zeta_k^*)\right], \quad (3.17)$$

for all $n \in \mathbb{N}$ satisfying $\tau_n < T$. We now prove $\alpha^* \in \mathcal{D}$ and assume on the contrary that $p := \mathbf{P}(\tau_n^* < T,\ \forall n \in \mathbb{N}) > 0$. Combining **(H3)**, (3.8) and (3.17), we derive

$$Y_0(0,i_0,x_0) \leq \bar{\psi}T + \mathbf{E}\left[\sup_{s\leq T}\left|Y_s^{\tau_n^*,\zeta_n^*,X_{\tau_n^*}^*}\right|\right] - n\bar{c}\mathbf{P}(\tau_k^* < T\ ,\ \forall k \geq 0) \leq \bar{\psi}T + \bar{Y} - n\bar{c}p\,.$$

Sending n to $-\infty$ leads to $Y_0^{0,i_0,x_0} = -\infty$ which contradicts $Y^{0,i_0,x_0} \in \mathcal{S}^2_{\mathbb{F}}$. Therefore $\mathbf{P}(\tau_k^* < T\ ,\ \forall k \geq 0) = 0$ i.e. $\alpha^* \in \mathcal{D}$. Finally, taking the limit as $n \to \infty$ in (3.17) leads to $Y_0^{0,i_0,x_0} = J(\alpha^*)$.

Step2. *The strategy α^* is optimal.*
According to Proposition 3.1, it suffices to consider finite strategies and we pick any $\alpha = (\tau_n,\zeta_n)_{n\geq 0} \in \mathcal{D}$. Since τ_1^* is optimal, we deduce from Part (i) of Proposition 3.2 that

$$\begin{aligned}
Y_0^{0,i_0,x_0} &- \mathbf{E}\left[\int_0^{\tau_1}\psi(s,i_0,X_s^{0,i_0,x_0})ds\right] \\
&\geq \mathbf{E}\left[\mathcal{O}_{\tau_1}^{0,i_0,x_0}\mathbf{1}_{\tau_1<T} + g(i_0,X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1=T}\right] \\
&\geq \mathbf{E}\left[(Y_{\tau_1}^{0,\zeta_1,x_0} + c(\tau_1,i_0,\zeta_1))\mathbf{1}_{\tau_1<T} + g(i_0,X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1=T}\right] \\
&\geq \mathbf{E}\left[\left(Y_{\tau_1}^{\tau_1,\zeta_1,X_{\tau_1}^{0,\zeta_1,x_0}} + c(\tau_1,i_0,\zeta_1)\right)\mathbf{1}_{\tau_1<T} + g(i_0,X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1<T}\right].
\end{aligned}$$

Proceeding as in step 1, an induction argument leads to

$$Y_0^{0,i_0,x_0} \geq \mathbf{E}\Big[\int_0^{\tau_n}\psi(i_0,X_s^\alpha)ds + Y_{\tau_n}^{\tau_n,\zeta_n,X_{\tau_n}^\alpha}\mathbf{1}_{\tau_n<T} + g(\zeta_n,X_T^\alpha)\mathbf{1}_{\tau_n=T} + \sum_{0<k\leq n} c(\tau_k,\zeta_{k-1},\zeta_k)\Big],$$

for all n satisfying $\tau_n < T$. Since the strategy α is finite, we deduce $Y_0^{0,i_0,x_0} \geq J(\alpha)$ by sending $n \to \infty$. □

3.3 Approximation by penalisation and link with constrained BSDEs with jumps

We finally conclude the paper and present in this paragraph the link between constrained Backward SDEs with jumps and optimal switching in non-Markovian cases.

Consider the constrained BSDE with jumps: find a quadruple $(Y, Z, U, K) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying

$$\begin{aligned} Y_t &= g(I_T, X^I_T) + \int_t^T \psi(s, I_s, X^I_s)ds + K_T - K_t \\ &\quad - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds, di), \quad 0 \le t \le T, \ a.s. \end{aligned} \tag{3.18}$$

with

$$- U_t(i) - c(t, I_{t^-}, i) \ \ge \ 0, \qquad d\mathbf{P} \otimes dt \otimes \lambda(di) \ \ a.e. \tag{3.19}$$

where the process (I, X^I) is defined by

$$\begin{aligned} I_t &= i_0 + \int_0^t \int_{\mathcal{I}} (i - I_{t^-})\mu(dt, di) \\ X^I_t &= x_0 + \int_0^t b(s, I_s, X^I_s)ds + \int_0^t \sigma(s, I_s, X^I_s)dWs \end{aligned}$$

The link between (3.18)-(3.19) and the optimal switching problem is given by the following result which extends the link between BSDEs with constrained jumps and BSDEs with oblique reflections presented in Proposition 2.2.

Proposition 3.3. *Under **(H3)**, **(H1)** holds for* (3.18)-(3.19) *and if we denote* (Y, Z, U, K) *its minimal solution we have*

$$Y_t = Y^{t, I_t, X^I_t}_t, \quad Z_t = Z^{t, I_{t^-}, X^I_t}_t \quad \text{ and } \quad U_t(i) = Y^{t, I_t, X^I_t}_t - Y^{t, I_{t^-}, X^I_t}_{t^-}. \tag{3.20}$$

In particular, we deduce $Y_0 = J(\alpha^*) = \sup_{\alpha \in \mathcal{A}} J(\alpha)$.

Proof. For any stopping time ν and any random variable η, let define the processes $(\tilde{Y}^{\nu,i,\eta,n}, \tilde{Z}^{\nu,i,\eta,n}, \tilde{K}^{\nu,i,\eta,n})_{i \in \mathcal{I}} \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^{\mathcal{I}}$ as the solution to the penalized BSDE

$$\begin{aligned} \tilde{Y}^{\nu,i,\eta,n}_t &= g(i, X^{\nu,i,\eta}_T) + \int_t^T \psi(s, i, X^{\nu,i,\eta}_s)ds - \int_t^T \langle \tilde{Z}^{\nu,i,\eta,n}_s, dWs\rangle \\ &\quad + n \int_t^T \Big\{ \sum_{j \in \mathcal{I}} [\tilde{Y}^{\nu,j,\eta,n}_s + c(s, i, j) - \tilde{Y}^{\nu,i,\eta,n}_s]^- \lambda(j) \Big\} ds \end{aligned}$$

Under **(H3)**, we know from Lemma 2.2 (or [11]) that $(Y^{\nu,i,\eta,n}, Z^{\nu,i,\eta,n}, K^{\nu,i,\eta,n})_{i \in \mathcal{I}}$ converges to $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i \in \mathcal{I}}$ as n goes to ∞, for each (ν, η). Proceeding as in the proof of Proposition 2.2, one easily checks that the quadruple $(\tilde{Y}^n, \tilde{Z}^n, \tilde{U}^n)$ defined by

$$\tilde{Y}^n_t = \tilde{Y}^{t, I_t, X^I_t, n}_t, \quad \tilde{Z}^n_t = \tilde{Z}^{t, I_{t^-}, X^I_t, n}_t \quad \text{ and } \quad \tilde{U}^n_t(i) = \tilde{Y}^{t, I_t, X^I_t, n}_t - \tilde{Y}^{t, I_{t^-}, X^I_t, n}_{t^-}$$

is solution to the penalized BSDE associated to (3.18)-(3.19), namely

$$\begin{aligned} Y_t &= g(I_T, X^I_T) + \int_t^T \psi(s, I_s, X^I_s)ds + n \int_t^T \int_{\mathcal{I}} [U_s(i) + c(s, I_{s^-}, i)]^- \lambda(di)ds \\ &\quad - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds, di), \quad 0 \le t \le T. \ a.s. \end{aligned}$$

From Proposition 3.2 (iii), we know that the monotone sequence $(\tilde{Y}^n)_n$ is bounded, and we derive from Remark 2.4 that Assumption **(H1)** is satisfied and that $(\tilde{Y}^n, \tilde{Z}^n, \tilde{U}^n)$ converges to (Y, Z, U), which concludes the proof. □

Remark 3.2. The optimal strategy can be described by the constrained BSDE with jumps (3.18)-(3.19). Indeed, using the definition of $(\tau_n^*, \zeta_n^*)_{n\geq 0}$ and the identification (3.20) we get

$$\tau_{n+1}^* \quad = \quad \inf\left\{ t \geq \tau_n^* \ ; \ \max_{j\in\mathcal{I}} \mathbf{E}\left[U_t(j) - c(t, \zeta_n^*, j) \Big| I_s = \zeta_n^* \quad \forall s \geq \tau_n^* \right] = 0 \right\}$$

and ζ_{n+1} *such that*

$$\mathbf{E}\left[U_{\tau_{n+1}^*}(\zeta_{n+1}^*) - c(\tau_{n+1}^*, \zeta_n^*, \zeta_{n+1}^*)] \Big| I_s = \zeta_n^* \quad \forall s \geq \tau_n \right] \quad = \quad 0.$$

4 Appendix

4.1 Viability property for BSDEs

We extend here the viability property of [4] for a closed convex cone $\mathcal{C}$ of $\mathbb{R}^m$. Let $(Y, Z) \in (\mathcal{S}_{\mathbb{F}}^{c,2} \times \mathbf{L}_{\mathbb{F}}^{\mathbf{2}}(\mathbf{W}))^m$ satisfying

$$Y_t \quad = \quad Y_T + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T \langle Z_s, dW_s \rangle + K_T - K_t$$

where $F \ : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ is a progressively measurable function satisfying **(H2)** (i) and (ii) and K is an $\mathbb{R}^m$-valued finite variation process such that

$$K_t \quad = \quad \int_0^t k_s d|K|_s \ ,$$

with $k_t \in \mathcal{C}$ and $|K|_s$ the variation of K on $[0, s]$. Denote $d_{\mathcal{C}}$ the distance to $\mathcal{C}$ (i.e. $d_{\mathcal{C}}(x) = \min_{y\in\mathcal{C}} |x - y|$) and $\Pi_{\mathcal{C}}$ the projection operator onto $\mathcal{C}$, then we have the following result:

Proposition 4.4. *Suppose* $Y_T \in \mathcal{C}$ *and there exists a constant* C^0 *such that* F *satisfies*

$$4\langle y - \Pi_{\mathcal{C}}(y), F(t, y, z)\rangle \quad \leq \quad \langle D^2 |d_{\mathcal{C}}|^2(y) z, z\rangle + 2C^0 |d_{\mathcal{C}}|^2(y) \tag{4.21}$$

for any point $y \in \mathbb{R}^m$ *where* $|d_{\mathcal{C}}|^2$ *is twice differentiable. Then, we have*

$$Y_t \quad \in \quad \mathcal{C}\,, \qquad \textit{for all} \ \ t \in [0,T] \quad \mathbf{P} - a.s.$$

Proof. Let $\eta \in C^\infty(\mathbb{R}^d)$ be a non-negative function with support in the unit ball and such that $\int_{\mathbb{R}^d} \eta(x) dx = 1$. For $\delta > 0$ and $x \in \mathbb{R}^d$ we put

$$\begin{aligned} \eta_\delta(x) \quad &:= \quad \frac{1}{\delta^d} \eta\Big(\frac{x}{\delta}\Big) \\ \phi_\delta(x) \quad &:= \quad |d_{\mathcal{C}}|^2 \star \eta_\delta(x) \quad = \quad \int_{\mathbb{R}^d} |d_{\mathcal{C}}(x - x')|^2 \eta_\delta(x') dx' \end{aligned}$$

Notice (see part (b) of the proof of Theorem 2.5 in [4]) that $\phi_\delta \in C^\infty(\mathbb{R}^d)$ and

$$\left\{\begin{array}{lcl} 0 \ \le \ \phi_\delta(x) & \le & (d_{\mathcal{C}}(x)+\delta)^2 \\ D\phi_\delta(x) & = & \int_{\mathbb{R}^d} D|d_{\mathcal{C}}(x')|^2\eta_\delta(x-x')dx' \ \text{ and } |D\phi_\delta(x)| \ \le \ 2(d_{\mathcal{C}}(x)+\delta) \\ D^2\phi_\delta(x) & = & \int_{\mathbb{R}^d} D^2|d_{\mathcal{C}}(x')|^2\eta_\delta(x-x')dx' \text{ and } \ 0 \ \le \ |D^2\phi_\delta(x)| \ \le \ 2I_d \end{array}\right. \tag{4.22}$$

Applying Itô's formula to $\phi_\delta(Y_t)$, this leads to

$$\begin{aligned} \mathbf{E}\phi_\delta(Y_t) &= \mathbf{E}\phi_\delta(Y_T) + \mathbf{E}\int_t^T \langle D\Phi_\delta(Y_s), F(s,Y_s,Z_s)\rangle ds - \frac{1}{2}\mathbf{E}\int_t^T \langle \Phi_\delta(Y_s)Z_s, Z_s\rangle ds \\ &\quad +\mathbf{E}\int_t^T \langle D\Phi_\delta(Y_s), k_s\rangle d|K|_s \\ &\le \delta^2 + \mathbf{E}\int_t^T\int_{\mathbb{R}^d} \Big[\langle D|d_{\mathcal{C}}(y)|^2, F(s,y,Z_s)\rangle - \frac{1}{2}\langle D^2|d_{\mathcal{C}}(y)|^2 Z_s, Z_s\rangle\Big]\eta_\delta(Y_s-y)dyds \\ &\quad -\mathbf{E}\int_t^T\int_{\mathbb{R}^d} \langle D|d_{\mathcal{C}}(y)|^2, F(s,y,Z_s)-F(s,Y_s,Z_s)\rangle \eta_\delta(Y_s-y)dyds \\ &\quad +\mathbf{E}\int_t^T\int_{\mathbb{R}^d} \langle D|d_{\mathcal{C}}(y)|^2, k_s\rangle \eta_\delta(Y_s-y)dyd|K|_s\,, \end{aligned}$$

for $0 \le t \le T$, $\delta > 0$. Since $k_t \in \mathcal{C}$ and $\mathcal{C}$ is a closed convex cone, we have $\langle D|d_{\mathcal{C}}(y)|^2, k_s\rangle$ ≤ 0. Then, combining (4.21) with inequality $2d_c(.) \le 1 + d_c(.)^2$, we get

$$\begin{aligned} \mathbf{E}\phi_\delta(Y_t) &\le \delta^2 + C^0\mathbf{E}\int_t^T\int_{\mathbb{R}^d} |d_{\mathcal{C}}(y)|^2\eta_\delta(y-Y_s)dyds \\ &\quad +2\mathbf{E}\int_t^T\int_{\mathbb{R}^d} d_{\mathcal{C}}(y)\eta_\delta(Y_s-y) \max_{y' :\, |y'-Y_s|\le\delta} |F(s,y',Z_s)-F(s,Y_s,Z_s)|dyds \\ &\le \delta^2 + C^0\int_t^T \mathbf{E}\phi_\delta(Y_s)ds + \mathbf{E}\int_t^T (1+\phi_\delta(Y_s)) \max_{y' :\, |y'-Y_s|\le\delta} |F(s,y',Z_s)-F(s,Y_s,Z_s)|ds\,. \end{aligned}$$

Using the Lipschitz property of F, we deduce

$$\mathbf{E}\phi_\delta(Y_t) \ \le \ C(\delta^2 + \delta + \int_t^T \mathbf{E}\phi_\delta(Y_s)ds),$$

and Gronwall's lemma leads to

$$\mathbf{E}\phi_\delta(Y_t) \ \le \ C(\delta^2+\delta), \quad 0 \le t \le T\ , \ \delta > 0\,.$$

Finally, from Fatou's Lemma, we have

$$\mathbf{E}|d_{\mathcal{C}}(Y_t)|^2 \ \le \ \liminf_{\delta\to 0} \mathbf{E}\phi_\delta(Y_t) \ = \ 0\ , \quad 0 \le t \le T\ ,$$

which concludes the proof. □

4.2 A multi-dimentional comparison theorem for BSDEs

We now turn to the obtention of a multi-dimentional comparison result. Consider (Y^1, Z^1, K) $\in (\mathcal{S}^{c,2}_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^m$ satisfying

$$Y^1_t \ = \ Y^1_T + \int_t^T F_1(s, Y^1_s, Z^1_s)ds - \int_t^T \langle Z^1_s, dW_s\rangle + K_T - K_t$$

and $(Y^2, Z^2) \in (\mathcal{S}^{c,2}_{\mathbb{F}} \times \mathbf{L^2_{\mathbb{F}}(W)})^m$ satisfying

$$Y^2_t \;=\; Y^2_T + \int_t^T F_2(s, Y^2_s, Z^2_s)ds - \int_t^T \langle Z^2_s, dW_s\rangle$$

Then we have the following comparison theorem generalizing the one in [10].

Theorem 4.2. *Suppose that $Y^1_T \geq Y^2_T$ and that, for any $(y, y', z, z') \in [\mathbb{R}^m]^2 \times [\mathbb{R}^{m\times d}]^2$, we have*

$$-4\langle y^-, F_1(t, y^+ + y', z) - F_2(t, y', z')\rangle \;\leq\; 2\sum_{i=1}^m \mathbf{1}_{y_i<0}|z_i - z'_i|^2 + 2C^0|y^-|^2 \;\mathbf{P} - a.s. \quad (4.23)$$

where $C^0 > 0$ is a constant. Then $Y^1_t \geq Y^2_t$, for all $t \in [0, T]$.

Proof. As in Theorem 2.1 in [10], it suffices to remark that F^1 is Lipschitz and apply Proposition 4.23 to the couple $(Y^1 - Y^2, Y^2)$ and the closed convex cone $\mathcal{C} = (\mathbb{R}^+)^m \times \mathbb{R}^m$. □

4.3 Monotonic Limit theorem for BSDE with jumps

This paragraph is devoted to the extension of Peng's monotonic limit theorem to the framework of BSDEs driven by a Brownian motion and a Poisson random measure.

Theorem 4.3. *Let $(Y^n, Z^n, U^n, K^n)_n$ be a sequence in $\mathcal{S}^2_{\mathbb{G}} \times \mathbf{L^2_{\mathbb{G}}(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2_{\mathbb{G}}}$ satisfying*

$$Y^n_t \;=\; Y^n_T + \int_t^T f(s, Y^n_s, Z^n_s, U^n_s)ds - \int_t^T \langle Z^n_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U^n_s(i)\mu(di, ds) + K^n_T - K^n_t,$$

for all $t \in [0, T]$. If $(Y^n)_n$ converges increasingly to Y with $\mathbf{E}[\sup_{t\in[0,T]} |Y_t|^2] < \infty$, then $Y \in \mathcal{S}^2_{\mathbb{G}}$ (up to a modification) and there exists $(Z, U, K) \in \mathbf{L^2_{\mathbb{G}}(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A^2_{\mathbb{G}}}$ such that

$$Y_t \;=\; Y_T + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(di, ds) + K_T - K_t,$$

for all $t \in [0, T]$. Moreover (Z, U) is the weak (resp. strong) limit of the sequence $(Z^n, U^n)_n$ in $\mathbf{L^2_{\mathbb{G}}(W)} \times \mathbf{L^2}(\tilde{\mu})$ (resp. $\mathbf{L^p_{\mathbb{G}}(W)} \times \mathbf{L^p}(\tilde{\mu})$, for $p < 2$). Finally, K is the weak limit of $(K^n)_n$ in $\mathbf{L^2_{\mathbb{G}}(0, T)}$ and, for any $t \in [0, T]$, K_t is the weak limit of $(K^n_t)_n$ in $\mathbf{L^2(\Omega, \mathcal{F}_t, P)}$.

Proof. The proof of Theorem 4.3 is an adaptation of the one presented in [15] and is performed in four steps.

1. *Uniform estimate.* Applying Itô's formula to $|Y^n|^2$ and using standard arguments (BDG inequality and Gronwall's Lemma) we get the existence of a constant $C > 0$ such that

$$\|Y^n\|_{\mathcal{S}^2_{\mathbb{G}}} + \|Z^n\|_{\mathbf{L^2_{\mathbb{G}}(W)}} + \|U^n\|_{\mathbf{L^2}(\tilde{\mu})} + \|K^n\|_{\mathcal{S}^2_{\mathbb{G}}} \;\leq\; C\,, \qquad n \in \mathbb{N}\,. \quad (4.24)$$

2. *Weak convergence.* Using the previous uniform estimate and the Hilbert structure of $\mathbf{L}^2_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{0}, \mathbf{T}) \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{0}, \mathbf{T})$, we deduce the existence of a subsequence of $(Z^n, U^n, K^n, f(., Y^n, Z^n, U^n))_n$, which converges weakly to some (Z, U, K, F). Identifying the limits of $(Y^n)_n$ and $(Z^n, U^n, K^n, f(., Y^n, Z^n, U^n))_n$, we get

$$Y_t = Y_T + \int_t^T F_s ds - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(di, ds) + K_T - K_t, \quad 0 \le t \le T. \tag{4.25}$$

3. *Properties of the process K.* We first observe from Lemma 2.2 in [15] that the process K admits a càdlàg modification. We then establish that the contribution of the jumps of K is mainly concentrated within a finite number of intervals with sufficiently small total length. This result is derived with similar arguments as in [15], relying only the right continuity of the filtration and the predictability of the process K. As in Lemma 2.3 in [15], observe also that, for any $\delta, \epsilon > 0$, there exists a finite number of pairs of stopping times $(\sigma_k, \tau_k)_{0 \le k \le N}$ with $0 < \sigma_k \le \tau_k \le T$ such that

(i) $(\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \emptyset$ for $j \neq k$;

(ii) $\mathbf{E} \sum_{k=0}^N (\tau_k - \sigma_k) \ge T - \varepsilon$;

(iii) $\mathbf{E} \sum_{k=0}^N \sum_{\sigma_k < t \le \tau_k} |\Delta K_t|^2 \le \delta$.

4. *Strong convergence.* Following the arguments of the proof of Theorem 3.1 in [12], we deduce the convergence of $(Z^n, U^n)_n$ to (Z, U) in $dt \otimes d\mathbf{P}-$measure. Together with the uniform estimate (4.24), this leads to the strong convergence of $(Z^n, U^n)_n$ to (Z, U) in $\mathbf{L}^{\mathbf{p}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{p}}(\tilde{\mu})$, $p < 2$. Combining the Lipschitz property of f with (4.25), we conclude that (Y, Z, U, K) satisfies

$$\begin{aligned} Y_t &= Y_T + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T \langle Z_s, dW_s \rangle \\ &\quad - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(di, ds) + K_T - K_t, \quad 0 \le t \le T. \end{aligned}$$

□

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